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Differentiation

Learning outcomes

In this Workbook you will learn what a derivative is and how to obtain the derivative of many commonly occurring functions. You will learn of the relationship between a derivative and the tangent line to a curve. You will learn something of the limiting process which arises in many areas of mathematics. You will learn how to use a table of derivatives to obtain the derivative of simple combinations of functions. Finally, you will learn how to take higher derivatives

Introducing Differentiation

11.1

Introduction

Differentiation is a technique which can be used for analysing the way in which functions change. In particular, it measures how rapidly a function is changing at any point. In engineering applications the function may, for example, represent the magnetic field strength of a motor, the voltage across a capacitor, the temperature of a chemical mix, and it is often important to know how quickly these quantities change.

In this Section we explain what is meant by the gradient of a curve and introduce differentiation as a method for finding the gradient at any point.

Prerequisites

Before starting this Section you should ...

- understand functional notation, e.g. $y = f(x)$
- be able to calculate the gradient of a straight line

Learning Outcomes

On completion you should be able to ...

- explain what is meant by the tangent to a curve
- explain what is meant by the gradient of a curve at a point
- calculate the derivative of a number of simple functions from first principles

1. Drawing tangents

Look at the graph shown in Figure 1a. A and B are two points on the graph, and they have been joined by a straight line. The straight line segment AB is known as a **chord**. We have lengthened the chord on both sides so that it extends beyond both A and B.

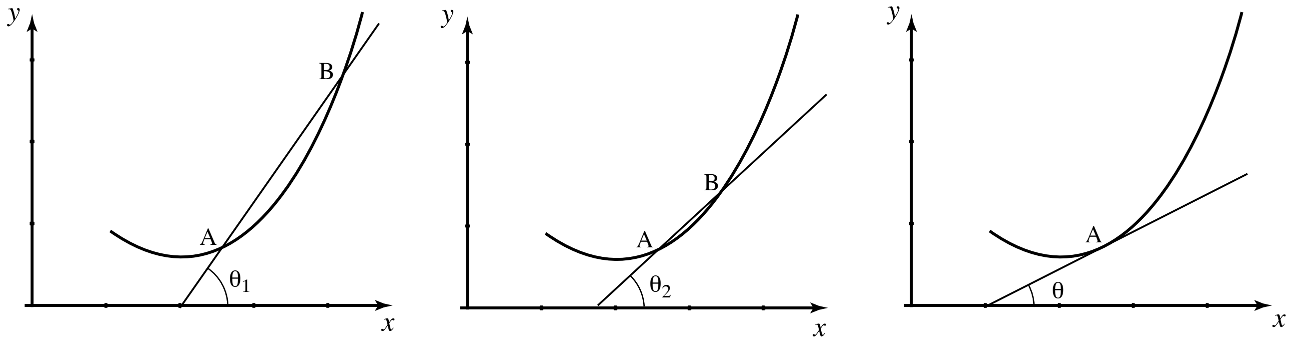
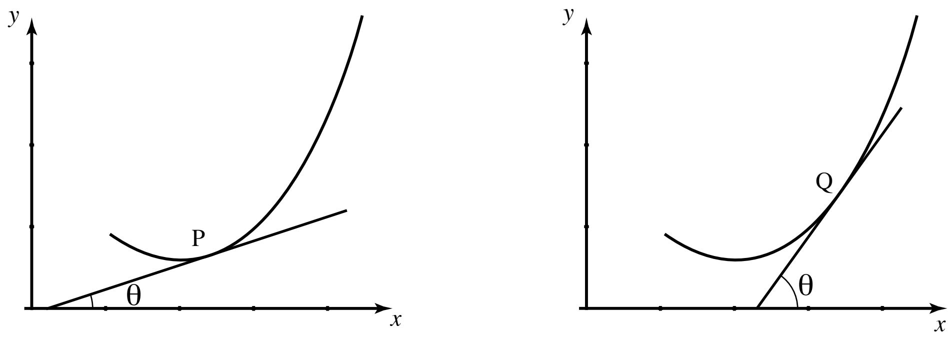


Figure 1a **Figure 1b** **Figure 1c**

In Figure 1b we have moved point B nearer to point A before drawing the extended chord. Imagine what would happen if we continue moving B nearer and nearer to A. You can do this for yourself by drawing additional points on the graph. Eventually, when B coincides with A, the extended chord is a straight line which just touches the curve at A. This line is now called the **tangent** to the curve at A, and is shown in Figure 1c.

If we know the position of two points on the line we can find the gradient of the straight line and can calculate the gradient of the tangent. We define the **gradient of the curve** at A to be the **gradient of the tangent** there. If this gradient is large at a particular point, the rate at which the function is changing is large too. If the gradient is small, the rate at which the function is changing is small. This is illustrated in Figure 2. Because of this, the gradient at A is also known as **the instantaneous rate of change** of the curve at A. Recall from your knowledge of the straight line, that if the line slopes upwards as we look from left to right, the gradient of the line is positive, whereas if the line slopes downwards, the gradient is negative.



The gradient of the tangent at P is small, so the rate at which the function is changing is small.

The gradient of the tangent at Q is large so the rate rate at which the function is changing is large.

Figure 2

Key Point 1

The gradient of the curve at a point, P, is equal to $\tan\theta$ where θ is the angle the tangent line at P makes with the positive x axis.

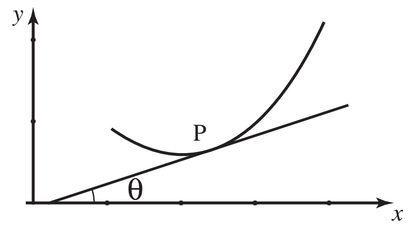
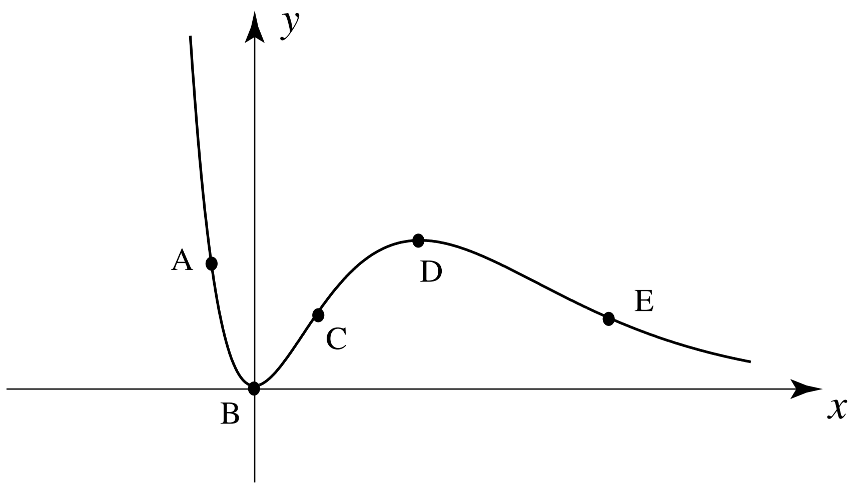


Figure 3

Task

Draw in, by eye, tangents to the curve shown below, at points A to E. State whether each tangent has positive, negative or zero gradient.



Your solution

Answer

A negative, B zero, C positive, D zero, E negative

In the following subsection we will see how to calculate the gradient of a curve precisely.

2. Finding the gradient at a specific point

In this subsection we shall consider a simple function to illustrate the calculation of a gradient. Look at the graph of the function $y(x) = x^2$ shown in Figure 4. Notice that the gradient of the graph changes as we move from point to point. In some places the gradient is positive; at others it is negative. The gradient is greater at some points than at others. In fact the gradient changes from point to point as we move along the curve.

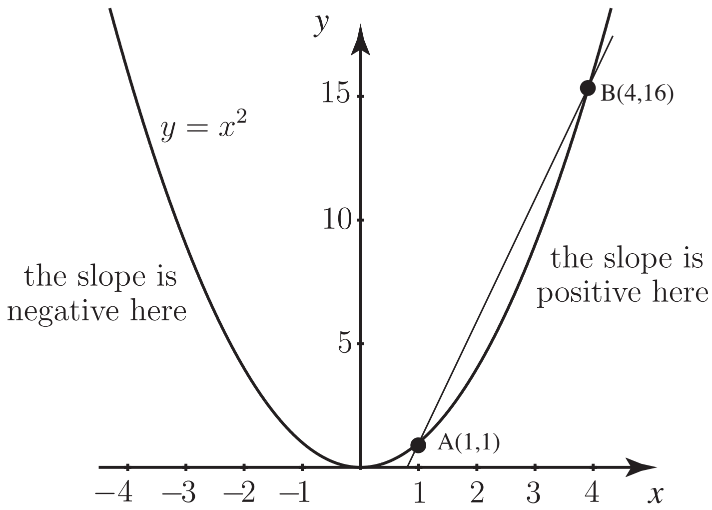


Figure 4

Inspect the graph carefully and make the following observations:

(a) A is the point with coordinates $(1, 1)$.
(b) B is the point with coordinates $(4, 16)$.
(c) We can calculate the gradient of the line AB from the formula

$$\text{gradient} = \frac{\text{difference between } y \text{ coordinates}}{\text{difference between } x \text{ coordinates}}$$

Therefore the gradient of chord AB is equal to $\dfrac{16-1}{4-1} = \dfrac{15}{3} = 5$. The gradient of AB is not the same as the gradient of the graph at A but we can regard it as an approximation, or estimate of the gradient at A. Is it an over-estimate or under-estimate ?

Add the point C to the graph in Figure 4 where C has coordinates $(3, 9)$. Draw the line AC and calculate its gradient.

Your solution

gradient =

Answer
$\dfrac{9-1}{3-1} = 4$. Would you agree that this is a better estimate of the gradient at A than using AB?

We now carry the last task further by introducing point D at $(2, 4)$ and point E at $(1.5, 2.25)$ as shown in Figure 5. The gradient of AD is found to be 3 and the gradient of AE is 2.5.

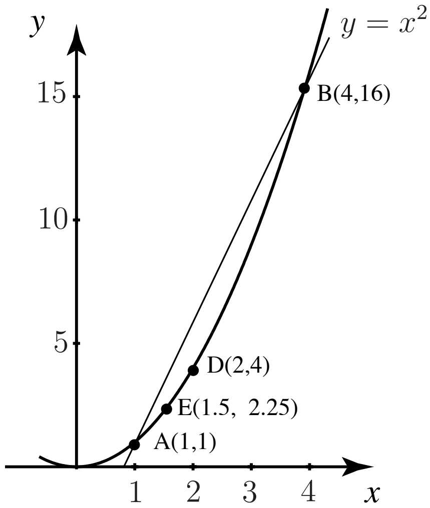


Figure 5

Observe that each time we carry out this procedure, and move the second point closer to A, the gradient of the line drawn is getting closer and closer to the gradient of the tangent at A. If we continue, the value we eventually obtain is the gradient of the tangent at A whose value is 2 as we will see shortly. This procedure illustrates how we define the gradient of the curve at A.

3. Finding the gradient at a general point

We now carry out the previous procedure more mathematically. Consider the graph of $y(x) = x^2$ in Figure 6. Let point A be any point with coordinates (a, a^2), and let point B be a second point with x coordinate $(a + h)$.

The y coordinate at A is a^2, because A lies on the graph $y = x^2$.

Similarly the y coordinate at B is $(a + h)^2$.

Therefore the gradient of the chord AB is

$$\frac{(a+h)^2 - a^2}{h}$$

This simplifies to

$$\frac{a^2 + 2ha + h^2 - a^2}{h} = \frac{2ha + h^2}{h} = \frac{h(2a+h)}{h} = 2a + h$$

This is the gradient of the line AB. As we let B move closer to A the value of h gets smaller and smaller and eventually tends to zero. We write this as $h \to 0$.

Now, as $h \to 0$, the gradient of AB tends to $2a$. Thus the gradient of the tangent to the curve at point A is $2a$. Because A is an arbitrary point, this result gives us a formula for finding the gradient of the graph of $y = x^2$ at any point: **the gradient is simply twice the x coordinate there**. For example when $x = 3$ the gradient is 2×3, that is 6, and when $x = 1$ the gradient is 2×1, that is 2 as we saw in the previous subsection.

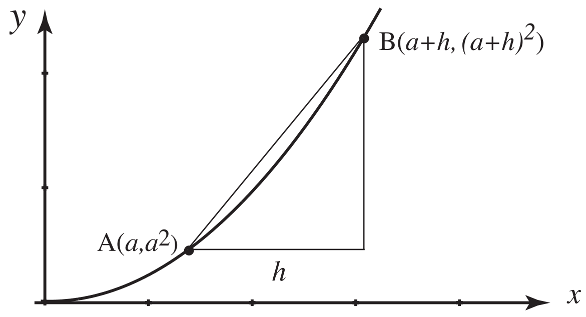


Figure 6

Generally, at a point whose coordinate is x the gradient is given by $2x$. The function, $2x$ which gives the gradient of $y = x^2$ is called the **derivative** of y with respect to x. It has other names too including the **rate of change** of y with respect to x.

A special notation is used to represent the derivative. It is not a particularly user-friendly notation but it is important to get used to it anyway. We write the derivative as $\dfrac{dy}{dx}$, pronounced 'dee y over dee x' or 'dee y by dee x' or even 'dee y, dee x'.

$\dfrac{dy}{dx}$ is not a fraction - so you can't do things like cancel the d's - just remember that it is the symbol or notation for the derivative. An alternative notation for the derivative is y'.

Key Point 2

The derivative of $y(x)$ is written $\dfrac{dy}{dx}$ or $y'(x)$ or simply y'

Exercises

1. Carry out the procedure above for the function $y = 3x^2$:
   (a) Let A be the point $(a, 3a^2)$.
   (b) Let B be the point $(a + h, 3(a + h)^2)$.
   (c) Find the gradient of the line AB.
   (d) Let $h \to 0$ to find the gradient of the curve at A.

2. Carry out the procedure above for the function $y = x^3$:
   (a) Let A be the point (a, a^3).
   (b) Let B be the point $(a + h, (a + h)^3)$.
   (c) Find the gradient of the line AB.
   (d) Let $h \to 0$ to find the gradient of the curve at A.

Answers

1. gradient $AB = 6a + 3h$, gradient at $A = 6a$. So, if $y = 3x^2$, $\dfrac{dy}{dx} = 6x$,

2. gradient $AB = 3a^2 + 3ah + h^2$, gradient at $A = 3a^2$. So, if $y = x^3$, $\dfrac{dy}{dx} = 3x^2$.

4. Differentiation of a general function from first principles

Consider the graph of $y = f(x)$ shown in Figure 7.

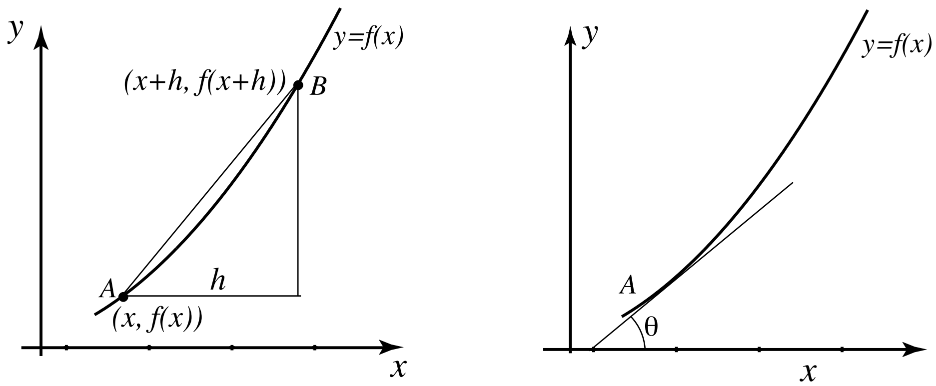


Figure 7: As $h \to 0$ the chord AB becomes the tangent at A

Carefully make the following observations:

(a) Point A has coordinates $(x, f(x))$.

(b) Point B has coordinates $(x + h, f(x + h))$.

(c) The straight line AB has gradient

$$\frac{f(x+h) - f(x)}{h}$$

(d) If we let $h \to 0$ we can find the gradient of the graph of $y = f(x)$ at the arbitrary point A, provided we can evaluate the appropriate limit on h. The resulting limit is the **derivative** of f with respect to x and is written $\dfrac{df}{dx}$ or $f'(x)$.

Key Point 3

Definition of Derivative

Given $y = f(x)$, its derivative is defined as

$$\frac{df}{dx} = \frac{f(x+h) - f(x)}{h} \qquad \text{in the limit as } h \text{ tends to } 0.$$

This is written

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In a graphical context, the value of $\frac{df}{dx}$ at A is equal to $\tan\theta$ which is the tangent of the angle that the gradient line makes with the positive x-axis.

Example 1

Differentiate $f(x) = x^2 + 2x + 3$ from first principles.

Solution

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{[(x+h)^2+2(x+h)+3]-[x^2+2x+3]}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{x^2+2xh+h^2+2x+2h+3-x^2-2x-3]}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{2xh+h^2+2h}{h}\right\}\\
&= \lim_{h\to 0}\{2x+h+2\}\\
&= 2x+2
\end{aligned}$$

Exercises

1. Use the definition of the derivative to find $\frac{df}{dx}$ when

   (a) $f(x) = 4x^2$, (b) $f(x) = 2x^3$, (c) $f(x) = 7x + 3$, (d) $f(x) = \frac{1}{x}$.
   (Harder: try (e) $f(x) = \sin x$ and use the small angle approximation $\sin\theta \approx \theta$ if θ is small and measured in radians.)

2. Using your results from Exercise 1 calculate the gradient of the following graphs at the given points:
   (a) $f(x) = 4x^2$ at $x = -2$, (b) $f(x) = 2x^3$ at $x = 2$, (c) $f(x) = 7x + 3$ at $x = -5$,
   (d) $f(x) = \frac{1}{x}$ at $x = 1/2$.

3. Find the rate of change of the function $y(x) = \frac{x}{x+3}$ at $x = 3$ by considering the interval $x = 3$ to $x = 3 + h$.

Answers

1. (a) $8x$, (b) $6x^2$, (c) 7, (d) $-1/x^2$, (e) $\cos x$.
2. (a) -16, (b) 24, (c) 7, (d) -4
3. $1/12$